Aalto University
School of Science

Exploiting Linear Hull in Matsui's Algorithm 1

Andrea Röck and Kaisa Nyberg

Department of Information and Computer Science
Aalto University, School of Science

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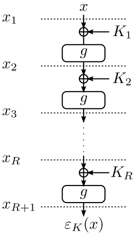
Outline

Introduction

Linear Cryptanalysis [Matsui 1994]

- Key-alternating iterated block cipher (R rounds):
  - Block size: n bits
  - Plain text: $x = x_1$
  - Key schedule: $K \mapsto K_1, \ldots, K_R \quad (K \in \mathbb{Z}_2^{\ell})$
  - Round function: $x_{i+1} = g(x_i \oplus K_i)$
  - Cipher text: $\varepsilon_K(x) = x_{R+1}$



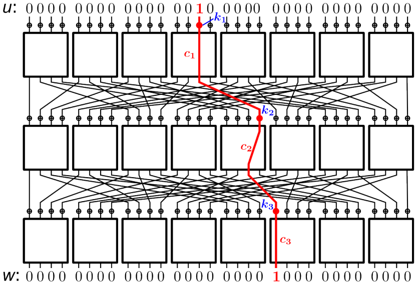
- Correlation over R rounds:

$$c_R(u, w, K) = \frac{\#\{u \cdot x = w \cdot \varepsilon_K(x)\} - \#\{u \cdot x \neq w \cdot \varepsilon_K(x)\}}{2^n}$$

- **Matsui's Algorithm 1**:
  - Use key dependency of $c_R(u, w, K)$ to learn $K \cdot v$
- **Matsui's Algorithm 2**:
  - Use that $|c_{R-1}(u, w, K)| > 0$ to gain information on K_R

Example 1

- Single strong trail (like in SERPENT)



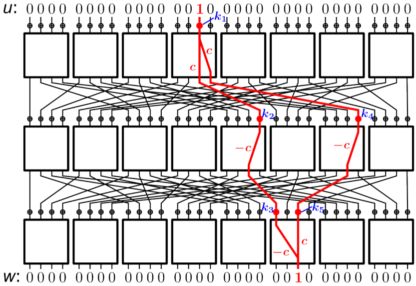
- **Piling-up Lemma** [Matsui 1994]

$$c(u, w, K) = (-1)^{k_1 \oplus k_2 \oplus k_3} c_1 c_2 c_3$$

Sign of trail-correlation depends on linear combination of key bits

Example 2 - Linear Hull

- Multiple strong trails (like in AES, PRESENT)



- The total correlation is the sum of the trail-correlations [Nyberg 2001, Deamen and Rijmen 2002]

$$c(u, w, K) = (-1)^{k_1 \oplus k_2 \oplus k_3} c^3 + (-1)^{k_1 \oplus k_4 \oplus k_5} (-c^3)$$

Linear Hull - Algorithm 2

- The average squared correlation of the linear approximation taken over all keys is equal to the sum of all squared trail correlations [Nyberg 1995]
- On average $|c_{R-1}(u, w, K)|$ is large enough to learn K_R
- For some keys, $|c_{R-1}(u, w, K)|$ is very small and the attack does not work [Murphy 2009]

Linear Hull - Algorithm 1

- Until now not analyzed
- Example: Two (independent) trails with trail-correlation c
  - For $1/4$ of keys: $c(u, w, K) = -2c$
  - For $1/2$ of keys: $c(u, w, K) = 0$ (Alg. 2 does not work)
  - For $1/4$ of keys: $c(u, w, K) = 2c$
- Correlation gives information of the key
  - In example: we learn 1.5 bits of information

Direct Attack

Idea

- Total correlation can be approximated by strong key-mask correlations: $c(u, w, K) \approx \sum_{v \in \mathcal{V}} \rho(v)(-1)^{v \cdot K}$
- Set of strong key masks: $\mathcal{V}$
- Key-mask correlation: $\rho(v)(-1)^{v \cdot K}$
- Possible correlations: $\mathcal{C} = \{c(u, w, K) : K \in \mathbb{Z}_2^\ell\}$
- Key classes: $\mathcal{K}(c) = \{K \in \mathbb{Z}_2^\ell : c(u, w, K) = c\}$
- **Goal**: For a given secret key K estimate $c \in \mathcal{C}$ from data such that $K \in \mathcal{K}(c)$

Efficient Precomputation

- How to compute $\mathcal{C}$ and $\mathcal{K}(c)$ faster than evaluating $\sum_{v \in \mathcal{V}} \rho(v)(-1)^{v \cdot K}$ for all $K \in \mathbb{Z}_2^{\ell}$?
- Let $t = \dim(\mathrm{span}(\mathcal{V}))$
- Can partition set of keys into 2^t disjoint subsets such that all the keys in a subset have the same correlation (subset $\subset \mathcal{K}(c)$ for a $c \in \mathcal{C}$)
- Use fast Walsh-Hadamard transform
- Precomputation complexities: time $\mathcal{O}\left(t2^t\right)$, memory $\mathcal{O}\left(2^t\right)$

Statistical Test

- $|\mathcal{C}|$-ary hypothesis testing problem: Find correct $c \in \mathcal{C}$
- $|\mathcal{K}(c)|$ varies a lot for different c
  - Use a priori probabilities $\pi_c = \Pr[c(u, w, K) = c]$ of c (Bayesian approach)
- Complexity depends on minimal distance in $\mathcal{C}$: $d = \min_{c_1 \neq c_2 \in \mathcal{C}} |c_1 - c_2|$
- Data complexity for error probability P_e

$$N = 8 \ln(2) \frac{\log_2(|\mathcal{C}| - 1) - \log_2 P_e}{d^2}$$

Gained Information

- **How much information do we learn?**
- Average learned information: **Shannon's entropy** of a priori probabilities π_c

$$h = -\sum_{c \in \mathcal{C}} \pi_c \log_2 \pi_c$$

- Special case: If all vectors in $\mathcal{V}$ linearly independent and $|\rho(v)| = const$: $c \in \mathcal{C}$ are binomial distributed and $\mathcal{O}\left(\frac{1}{2}\log_2(\frac{\pi e}{2}|\mathcal{V}|)\right)$
- Always $h \leq \log_2 |\mathcal{C}|$

Related Key Attack

Idea

- Complexity of direct attack increases with number of strong key masks $|\mathcal{V}|$
- Reduce number of relevant key masks by related key attack
- Correlation difference:

$$
\begin{aligned}
\Delta(K,\alpha) &= c(u,w,K) - c(u,w,K\oplus\alpha) \\
&= \sum_{v\in\mathcal{V}} (-1)^{v\cdot K}\rho(v) - \sum_{v\in\mathcal{V}} (-1)^{v\cdot(K\oplus\alpha)}\rho(v)
\end{aligned}
$$

- Reduced key mask set: $\mathcal{V}_\alpha = \{v\in\mathcal{V} : v\cdot\alpha = 1\}$

$$
\Delta(K,\alpha) = 2\sum_{v\in\mathcal{V}_\alpha} (-1)^{v\cdot K}\rho(v)
$$

- Statistical test and definition of $\mathcal{C}_\alpha, d_\alpha, t_\alpha, h_\alpha$ equivalent to direct attack

Multiple Related Key Attack

- For a given $\mathcal{V}$ we can learn at most $t = \dim(\mathrm{span}(\mathcal{V}))$ bits of information
- Independent case: all vectors in $\mathcal{V}$ are linearly independent
  - Given any $v \in \mathcal{V}$ choose α_v such that for all $v' \in \mathcal{V}$:

$$\alpha_v \cdot v' = \delta_{v,v'} = \begin{cases} 1 & \text{if } v' = v \\ 0 & \text{otherwise} \end{cases}$$

  - Then $\mathcal{V}_{\alpha_v} = \{v\}$ and from $\Delta(K, \alpha_v) = 2(-1)^{v \cdot K}\rho(v)$ we learn $K \cdot v$ (as in the classical Alg. 1)
  - Applying related key attacks for all α_v, $v \in \mathcal{V}$ gives us $|\mathcal{V}| = t$ bits of information
- Can be generalized to dependent case by considering a basis of $\mathrm{span}(\mathcal{V})$ instead of $\mathcal{V}$ to learn $\leq t$ bits

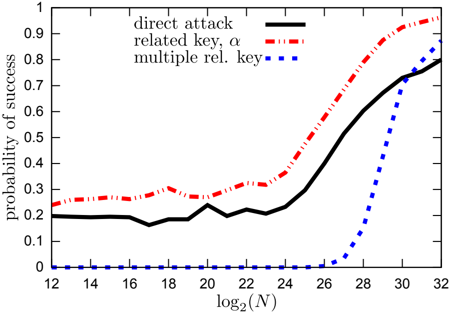
Results from Experiments

Round Reduced PRESENT [Bogdanov et al. 2007]

- 7 round 80-bit key version of PRESENT cipher
- Key schedule is semi-linear
- Extended key $K \in \mathbb{Z}_2^{104}$: round keys depend linearly on K
- Multiple strong trails of correlation 2^{-2R} for R rounds
- **Direct attack**
  - $|\mathcal{V}| = 24$, $|\mathcal{C}| = 13$, $t = 15$, $|\rho(v)| = 2^{-14}$, $h = 3.2$
- **Related key approach**
  - Assert that $K \oplus \alpha$ can be produced (α must not influences non-linear parts of the key schedule)
  - $|\mathcal{V}_\alpha| = 9$, $|\mathcal{C}_\alpha| = 10$, $t_\alpha = 9$, $|\rho(v)| = 2^{-14}$, $h_\alpha = 2.6$
- **Multiple related key approach**
  - Learn 14.25 bits of information
- 400 random keys and 2^{32} plain text blocks
- Direct attack theoretically applicable on up to 12 rounds for an 80-bit key and on up to 14 rounds for a 128-bit key

Probability of Success

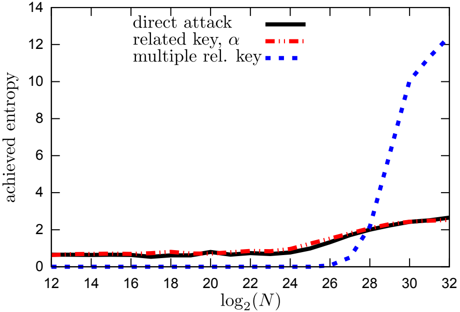
- Test for 400 different keys



- Multiple related key is only correct if all key classes are correct
- Related key has higher success probability

Achieved Entropy

- Achieved entropy: entropy $\times$ success probability
- Test for 400 different keys



- For $N \geq 2^{28}$ the multiple related key approach leads to best result

Conclusion

Comparison (1)

- **Algorithm 1 vs. Algorithm 2 for multiple strong trails**

Algorithm 1	**Algorithm 2**
Targets K	Targets K_R
Works for all keys	Works for most keys
Data complexity inverse proportional to minimal distance d between elements in $\mathcal{C}$	For about half of the keys the data complexity is better or equal to $\mathcal{O}\left(\left(\sum_{v\in\mathcal{V}} \rho(v)^2\right)^{-1}\right)$

Comparison (2)

- **Multiple related key approach vs. multidimensional linear cryptanalysis for Algorithm 1**

	Multiple related key	**Multidimensional**
Setting	One approximation with multiple strong trails	m linearly independent approx. each with one strong trail
Dim.	t dimension of trail set $\mathcal{V}$	m number of base approx.
Data N	$\mathcal{O}\left(\max\limits_{1\le i\le t} \frac{(\lvert\mathcal{C}_{\alpha_i}\rvert - 1) - \log P_e}{d^2_{\alpha_i}}\right)$	$\mathcal{O}\left(\frac{(2^m - 1) - \log P_e}{2^m \sum_{\eta\in\mathbb{Z}_2^m}(p_\eta - 2^{-m})^2}\right)$
Offline	t: $\mathcal{O}\left(t^2 2^t\right)$, m: $\mathcal{O}\left(t2^t\right)$	t: $\mathcal{O}\left(m2^m\right)$, m: $\mathcal{O}\left(2^m\right)$
Online	t: $\mathcal{O}\left(tN\right)$, m: $\mathcal{O}\left(t\right)$	t: $\mathcal{O}\left(mN\right)$, m: $\mathcal{O}\left(2^m\right)$
Inform.	$\sim t$ bits	m bits

Conclusion

- Application of Matsui's Algorithm 1 on key-alternating iterated block cipher which has linear approximations with multiple strong trails
- Precomputation complexity increases with number of trails
- Data complexity is inverse proportional to minimal distance between possible correlations
- Related key analysis reduces number of considered trails
- Several key differences can be combined for a better result