Characterizations of Bounded Semilinear Languages by One-way and Two-way Deterministic Machines

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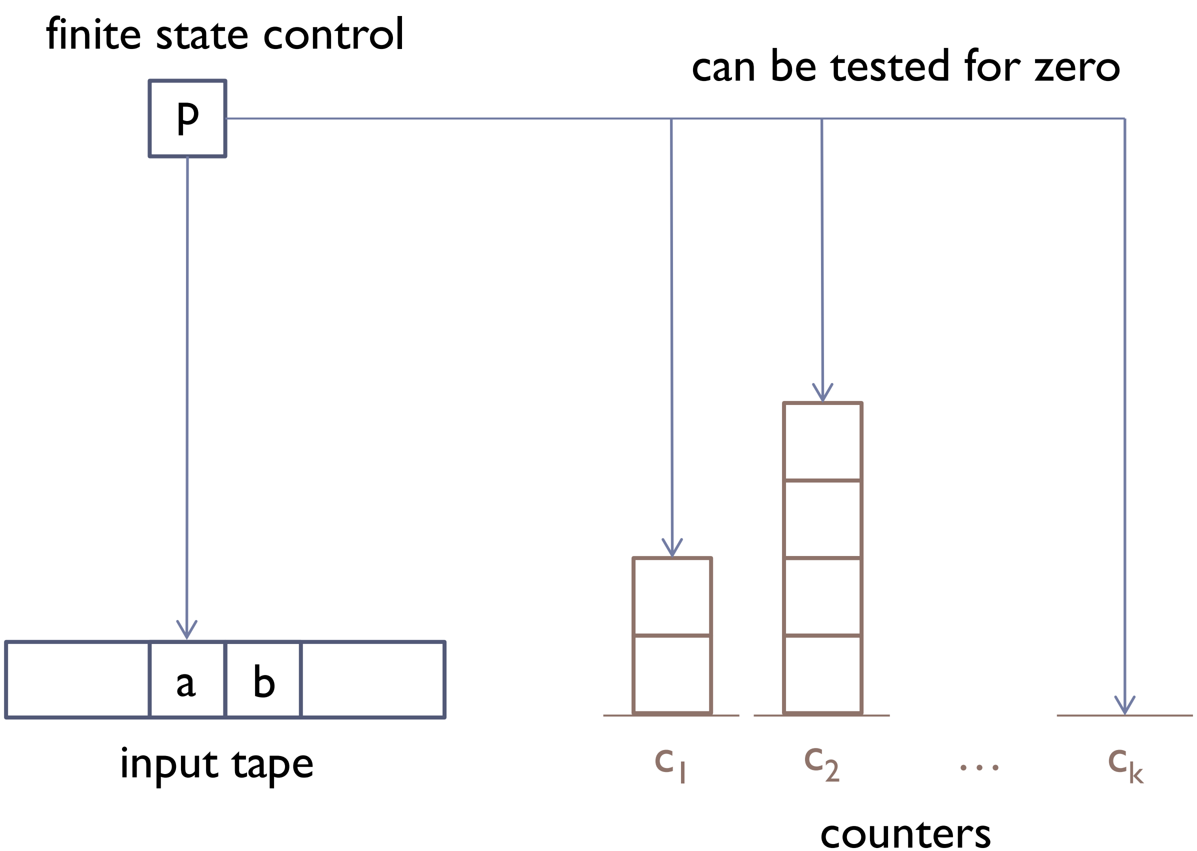
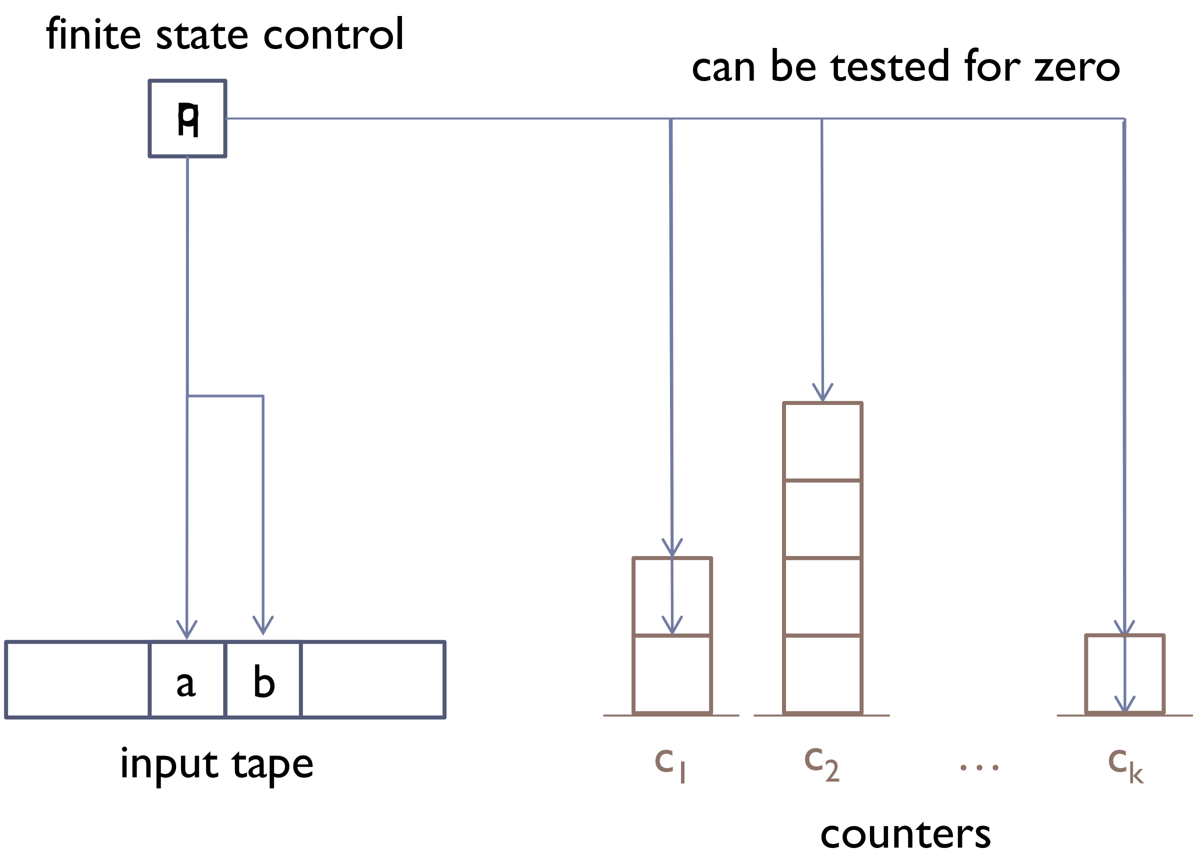
Conceptual diagram of a counter machine



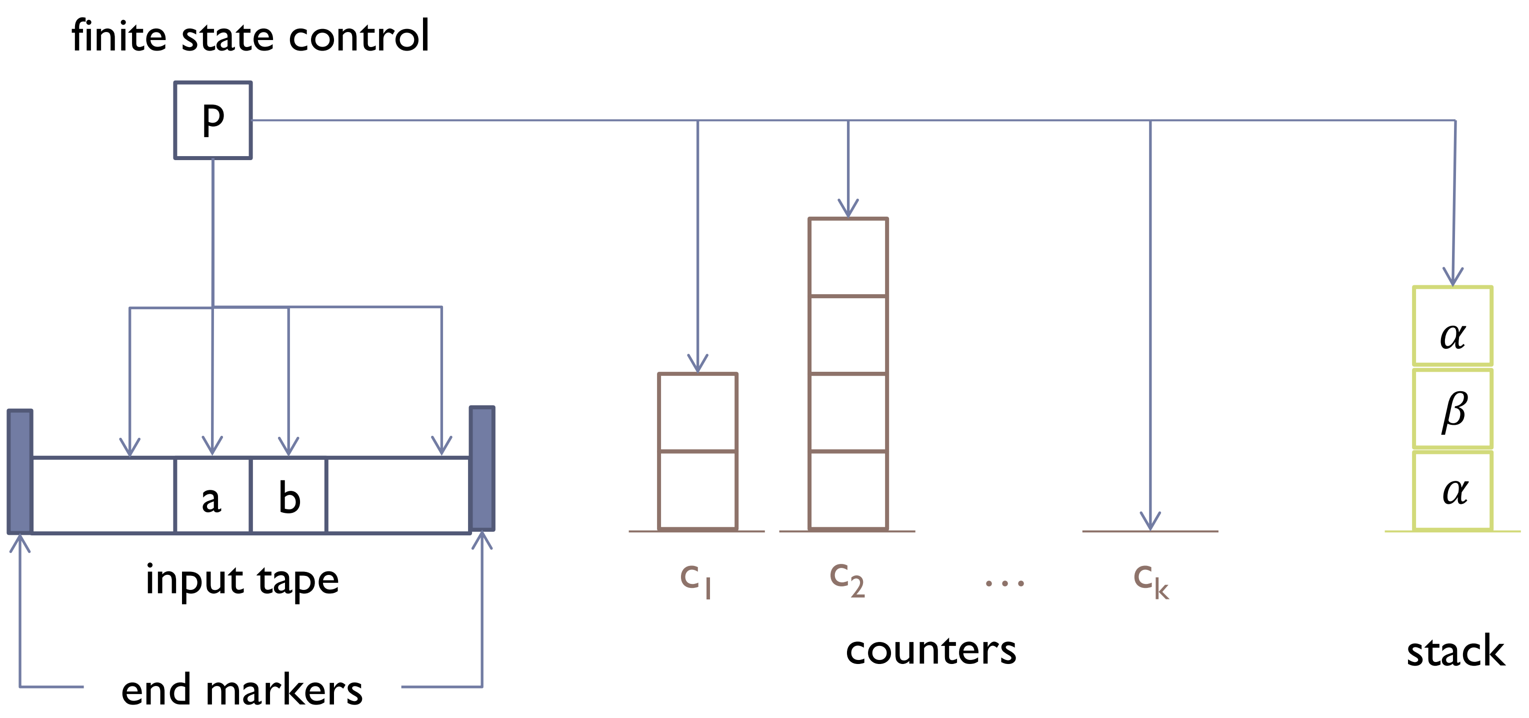
Conceptual diagram of a counter machine

$$\delta(p, a, 1, 1, \dots, 0) = (q, R, 0, -1, \dots, +1)$$



Variants of counter machine

- Pushdown counter machines
- 2-way counter machines



Formal definition of pushdown counter machines

For $k \geq 0$, a pushdown k-counter machine is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where

Q	a finite set of states
Σ, Γ	input and stack alphabets
Z_0	the bottom stack symbol
q_0	the initial state
F	a set of accepting states

and δ is a relation $Q \times \Sigma \times \Gamma \times \{0,1\}^k \rightarrow Q \times \{\mathrm{S,R}\} \times \Gamma^* \times \{-1, 0, +1\}^k$.

Reversal-bounded counter machines

A finite automaton augmented with 2 counters is Turing-complete. [Minsky 1961]

Definition (Equivalent*)

For a positive integer $k \geq 1$, a counter machine is r-reversal if for each of its counters, the number of alternations between non-decreasing mode and non-increasing mode – called reversal – is upper-bounded by r in any accepting computation.

*This equivalence needs a help of finite state control to remember how many reversals each counter has made.

r-reversal counter and 1-reversal counter

One can simulate a counter that makes r-reversals by counters that make 1 reversal.

Lemma

For any positive integer $r \geq 1$, an r-reversal counter can be simulated by $\left\lceil \frac{r+1}{2} \right\rceil$ 1-reversal counters.

This lemma enables us to focus on the 1-reversal counter machines.

Classes of counter machines

$\mathrm{DCM}(k)$	• the class of deterministic k-counters machines
$\mathrm{NCM}(k)$	• The class of (non-deterministic) k-counters machines
$\mathrm{DPCM}(k)$	• The class of deterministic pushdown k-counters machines
$\mathrm{NPCM}(k)$	• The class of pushdown k-counters machines
$\mathrm{2DCM}(k)$	• The class of 2-way deterministic k-counters machines

For a machine class M, by $\mathcal{L}(M)$, we denote the set of all languages accepted by a machine in M.

Classes of counter machines (cont.)

- $\mathrm{DFA} = \mathrm{DCM}(0)$
- $\mathrm{NFA} = \mathrm{NCM}(0)$
- $\mathrm{DPDA} = \mathrm{DPCM}(0)$
- $\mathrm{NPDA} = \mathrm{NPCM}(0)$
- $\mathrm{DCM} = \bigcup_{k \geq 0} \mathrm{DCM}(k)$ is the class of deterministic counter machines.

Note that $\mathcal{L}(\mathrm{DFA}) = \mathcal{L}(\mathrm{NFA}) \subsetneq \mathcal{L}(\mathrm{DPDA}) \subsetneq \mathcal{L}(\mathrm{NPDA})$.

Semilinear sets

A subset $Q \subseteq N^k$ is a linear set if there exist k-dimensional integer vectors $v_0, v_1, \dots, v_n$ such that

$$Q = \{v_0 + t_1 v_1 + \cdots + t_n v_n \mid t_1, \dots, t_n \in N\}.$$

A set is semilinear if it is a finite union of linear sets.

Bounded semilinear languages

A language $L \subseteq \Sigma^*$ is bounded if there exist $k \geq 1$ and $x_1, x_2, \ldots, x_k \in \Sigma^+$ such that

$$L \subseteq {x_1}^* {x_2}^* \cdots {x_k}^*.$$

If $x_1, x_2, \ldots, x_k$ are letters, L is especially said to be letter-bounded.

Definition

For a bounded language $L \subseteq {x_1}^* {x_2}^* \cdots {x_k}^*$, let

$$Q(L) = \{(i_1, \ldots, i_k) \mid {x_1}^{i_1} {x_2}^{i_2} \cdots {x_k}^{i_k} \in \mathrm{L}\}.$$

If this set is semilinear, then we say that L is bounded semilinear.

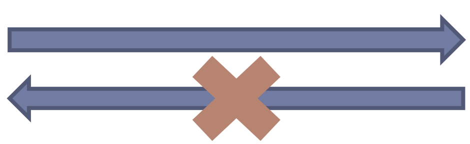
Finite-turn machines

Finite-turn

For an integer $t \geq 0$, a 2-way machine is t -turn if the machine can accept an input with at most t "turns" of the input head.
A 2-way machine is finite-turn if it is a t-turn for some $t \geq 0$.

Finite-crossing

For an integer $c \geq 1$, a 2-way machine is c -crossing if every accepted input admits a computation by the machine during which the input head crosses the boundary between any two adjacent symbols no more than c times.
A 2-way machine is finite-crossing if it is a c-turn for some $c \geq 1$.

Finite-turn  Finite-crossing

Main theorem

Main Theorem

The following 5 statements are equivalent for every bounded language L:

1. *L is bounded semilinear;*
2. *L can be accepted by a finite-crossing* $2\mathrm{DPCM}$*;*
3. *L can be accepted by a finite-turn* $2\mathrm{DPDA}$ *whose stack is reversal-bounded;*
4. *L can be accepted by a finite-turn* $2\mathrm{DCM}(1)$*;*
5. *L can be accepted by a* $1\mathrm{DCM}$.

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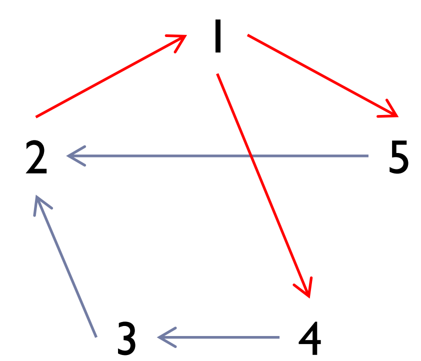
3. L *can be accepted by a finite-turn* $\mathrm{2DPDA}$ *whose stack is reversal-bounded;*
4. L *can be accepted by a finite-turn* $\mathrm{2DCM}(1)$*;*
5. L *can be accepted by a* $\mathrm{1DCM}$.

$4 \rightarrow 3 \rightarrow 2$ and $5 \rightarrow 2$ trivially hold since

- $\mathrm{finite{-}turn\ 2DCM(1)} \subseteq \mathrm{finite{-}turn\ 2DPDA} \subseteq \mathrm{finite{-}crossing\ 2DPCM}$, and
- $\mathrm{1DCM} \subseteq \mathrm{finite{-}crossing\ 2DPCM}$.

Let us prove

- $1 \rightarrow 4, 1 \rightarrow 5$
- $2 \rightarrow 1$



Proof direction for $1 \to 4, 1 \to 5$

Accepted by \ Language	$L \subseteq a_1^* \cdots a_k^*$ $Q(L)$ is semilinear	$L \subseteq x_1^* \cdots x_k^*$ $Q(L)$ is semilinear
1NCM	result 1 [Ibarra 78]	Use result 1 and non-deterministic guess.
finite−turn 2DCM(1) or 1DCM	Linear Diophantine-equations turn out to be solvable by these machines	$1 \to 4, 1 \to 5$

Letter-bounded to bounded

- Let $L \subseteq {x_1}^* \cdots {x_k}^*$ s.t. $Q(L)$ is semilinear.
- Given a word w,

$$w \in L \Leftrightarrow \exists (i_1, \ldots, i_k) \in Q(L), w = {x_1}^{i_1} \cdots {x_k}^{i_k}.$$

- Let $L' = \{ {a_1}^{i_1} \cdots {a_k}^{i_k} \mid (i_1, \ldots, i_k) \in Q(L) \}$, and construct a $1\mathrm{NCM}$ M' for it.

- $1\mathrm{NCM}$ M for L
  1. reads w;
  2. non-deterministically decomposes it as $w = {x_1}^{i_1} \cdots {x_k}^{i_k}$;
  3. runs M' on ${a_1}^{i_1} \cdots {a_k}^{i_k}$, and returns M''s decision as its decision for w and L.

Proof direction for $1 \to 4, 1 \to 5$

Language / Accepted by	$L \subseteq a_1{}^* \cdots a_k{}^*$ $Q(L)$ is semilinear	$L \subseteq x_1{}^* \cdots x_k{}^*$ $Q(L)$ is semilinear
1NCM	result I [Ibarra 78]	Use result I and non-deterministic guess.
finite−turn 2DCM(1) or 1DCM	Linear Diophantine-equations turn out to be solvable by these machines	$1 \to 4, 1 \to 5$

- How can a deterministic machine non-deterministic decompose an input?
- Our answer is "this decomposition does not require non-determinism."

Deterministic decomposition

- Let c be a sufficiently-large constant.
  - This is efficiently computable from a given language $L \subseteq {x_1}^* \cdots {x_k}^*$ s.t. $Q(L)$ is semilinear.
- Let $L' = L \cap {x_1}^{\geq c} {x_2}^{\geq c} \cdots {x_k}^{\geq c}$.

Theorem [Fine and Wilf 1965]

Let u, v be primitive words. If u^* and v^* share their prefix of length $|u| + |v| - \gcd(|u|, |v|)$, then $u = v$.

- Based on this theorem, we can figure out that looking-ahead by some constant distance on an input tape settles the decomposition.

Main theorem

Main Theorem

The following 5 statements are equivalent for every bounded language L:

1. *L is bounded semilinear;*

   ⇑

2. *L can be accepted by a finite-crossing* 2DPCM;
3. *L can be accepted by a finite-turn* 2DPDA *whose stack is reversal-bounded;*
4. *L can be accepted by a finite-turn* 2DCM(1);
5. *L can be accepted by a* 1DCM.

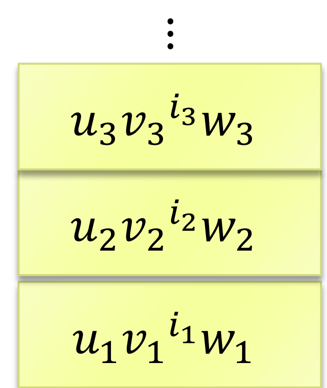
Proof for 2 → 1

Lemma

If a letter-bounded language is accepted by a finite-crossing 2DPDA M, *then it is accepted by a finite-crossing* 2DCM.

Proof idea

- In a computation by a finite-crossing 2DPDA, contents of M's stack consist of constant # of blocks of the form uv^iw (pumped structure).
- Each of these blocks correspond to a writing phase during which the stack is never popped.
- # of "long" writing phases is bounded by a constant c.
- We build a $2\mathrm{DCM}(c)$, and let its counters to store $i_1, i_2, i_3, \ldots$, and let its finite state control to remember $(u_1, v_1, w_1), \ldots$



stack

Proof for $2 \to 1$

The previous result is generalized for a broader class of finite-crossing 2DPCM.

Lemma

If a letter-bounded language is accepted by a finite-crossing 2DPCM, *then it is accepted by a finite-crossing* 2DCM.

Proof for $2 \rightarrow 1$

Theorem

A bounded language that is accepted by a finite-crossing $\mathrm{2DPCM}$ *is effectively semilinear.*

Proof.

Let M be a finite-crossing $\mathrm{2DPCM}(c)$ for some $c \geq 0$ s.t. $L(M) \subseteq {x_1}^* \cdots {x_k}^*$.

1. It is easy to construct a finite-crossing $\mathrm{2DPCM(c)}$ for $\{{a_1}^{i_1} \cdots {a_k}^{i_k} \mid {x_1}^{i_1} \cdots {x_k}^{i_k} \in L(M)\}$.
2. This language is letter-bounded, and hence, this machine can be converted into an equivalent finite-crossing $\mathrm{2DCM}$.
3. It is known that if a letter-bounded language is accepted by a finite-crossing $\mathrm{2NCM}$, then the language is bounded semilinear.
4. Thus, $Q(L(M))$ is semilinear.

Main theorem

Main Theorem (Proved!!)

The following 5 statements are equivalent for every bounded language L:

1. L *is bounded semilinear;*
2. L *can be accepted by a finite-crossing* $\mathrm{2DPCM}$*;*
3. L *can be accepted by a finite-turn* $\mathrm{2DPDA}$ *whose stack is reversal-bounded;*
4. L *can be accepted by a finite-turn* $\mathrm{2DCM}(1)$*;*
5. L *can be accepted by a* $\mathrm{1DCM}$*.*

Open

Is every bounded language accepted by a finite-crossing $\mathrm{2NPCM}$ bounded semilinear.

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