An Extension of the Lyndon Schützenberger Result to Pseudoperiodic Words

Elena Czeizler[1] Eugen Czeizler[1]
Lila Kari[2] Shinnosuke Seki[2]

[1]Department of IT, Åbo Akademi University

[2]Department of Computer Science, University of Western Ontario

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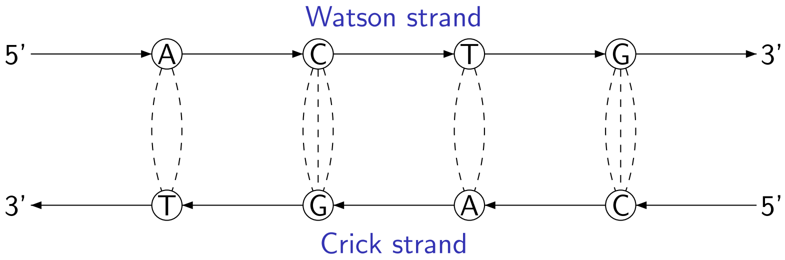
Informational equivalence between Watson and Crick strands



The Crick strand CAGT is obtained from the Watson strand ACTG by the antimorphic involution τ defined as $\tau(\mathtt{A}) = \mathtt{T}$, $\tau(\mathtt{T}) = \mathtt{A}$, $\tau(\mathtt{C}) = \mathtt{G}$, $\tau(\mathtt{G}) = \mathtt{C}$.

Observation

Two WK-complementary strands are equivalent w.r.t. the information they encode.

A mathematical model of Watson-Crick complementarity

Definition

A mapping $\theta : \Sigma^* \rightarrow \Sigma^*$ is called an antimorphic involution if

1. for any $x, y \in \Sigma^*$, $\theta(xy) = \theta(y)\theta(x)$ (antimorphism), and
2. θ^2 is the identity (involutive).

Actually,

$$\tau(\mathtt{ACTG}) = \tau(\mathtt{G})\tau(\mathtt{ACT}) = \cdots = \tau(\mathtt{G})\tau(\mathtt{T})\tau(\mathtt{C})\tau(\mathtt{A}) = \mathtt{CAGT}.$$

Observation

For an arbitrary antimorphic involution θ and a word $u \in \Sigma^*$, u and $\theta(u)$ are informationally equivalent.

θ-primitivity

A word $u \in \Sigma^+$ is primitive if for any $t \in \Sigma^+$, $u \in t^+$ implies $u = t$.

Definition ([CKS08])

A word $u \in \Sigma^+$ is said to be θ-primitive if for any $t \in \Sigma^+$, $u \in t\{t, \theta(t)\}^*$ implies $u = t$.

Definition ([CKS08])

The θ-primitive root of $u \in \Sigma^+$ is a θ-primitive word $t \in \Sigma^+$ s.t. $u \in t\{t, \theta(t)\}^*$.

The uniqueness of θ-primitive root is guaranteed.

An extended Fine and Wilf theorem

Let $u, v \in \Sigma^+$ and θ be an antimorphic involution.

Theorem ([CKS08])

If there exist $\alpha \in \{u, \theta(u)\}^*$ *and* $\beta \in \{v, \theta(v)\}^*$ *which share a prefix of length* $\mathrm{lcm}(|u|, |v|)$, *then* $u, v \in \{t, \theta(t)\}^+$ *for some* θ*-primitive word* $t \in \Sigma^+$.

Theorem ([CKS08])

If there exist $\alpha \in \{u, \theta(u)\}^*$ *and* $\beta \in \{v, \theta(v)\}^*$ *which share a prefix of length* $2\max(|u|, |v|) + \min(|u|, |v|) - \gcd(|u|, |v|)$, *then* $u, v \in \{t, \theta(t)\}^+$ *for some* θ*-primitive word* $t \in \Sigma^+$.

Lyndon-Schützenberger equation

For words $u, v, w \in \Sigma^*$, an equation of the form

$$u^\ell = v^n w^m$$

is called the Lyndon-Schützenberger equation originally proposed in [LySc62].

Theorem

If $\ell, n, m \geq 2$, then the above equation implies $u, v, w \in t^+$ for some primitive word $t \in \Sigma^+$.

Its concise proof is available at, e.g., [Lot83, HaNo04].

An extended Lyndon Schützenberger equation

We extend the LS-equation as follows:

$$u_1 \cdots u_\ell = v_1 \cdots v_n w_m \cdots w_1,$$

for $u, v, w \in \Sigma^+$ and $\ell, n, m \geq 2$, where $u_1, \ldots, u_\ell \in \{u, \theta(u)\}$, $v_1, \ldots, v_n \in \{v, \theta(v)\}$, and $w_1, \ldots, w_m \in \{w, \theta(w)\}$.

Problem

Find conditions on ℓ, n, m under which the exLS equation implies $u, v, w \in \{t, \theta(t)\}^+$ for some θ-primitive word $t \in \Sigma^+$.

When ℓ, n, m guarantees the existence of such t, we say that (ℓ, n, m) imposes θ-periodicity.

Cases when (ℓ, n, m) does not impose θ-periodicity

Proposition

If one of ℓ, n, m is 2, then (ℓ, n, m) does not impose θ-periodicity.

Example

Let θ be the mirror image on $\{a, b\}$. Let $u = a^k b^2 a^{2k}$, $v = (a^{2k} b^2 a^k)^\ell a^{2k} b^2$, and $w = a^2$ for some $k, \ell \geq 1$. Then $\theta(u)^{\ell+1} u^{\ell+1} = v^2 w^k$.
Actually, when $\ell = k = 2$, then

$$\begin{aligned} \theta(u)^2 u^2 &= (a^4 b^2 a^2)^2 (a^2 b^2 a^4)^2 \\ &= a^4 b^2 a^6 b^2 a^4 b^2 a^6 b^2 a^4 \\ &= (a^4 b^2 a^6 b^2)^2 (a^2)^2 = v^2 w^2 \end{aligned}$$

Observation

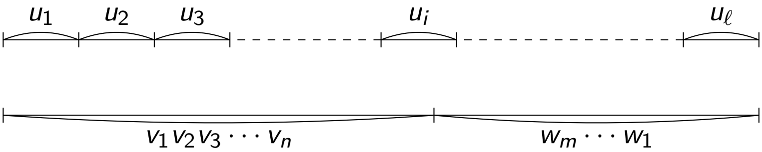
$\ell, n, m \geq 3$ must hold for (ℓ, n, m) to impose θ-periodicity.

An application of exFW theorem I

Many cases can be solved by applying the exFW theorem.

Proposition

$(\geq 6, \geq 3, \geq 3)$ *imposes θ-periodicity.*



Due to the symmetric roles of $v_1 \cdots v_n$ and $w_m \cdots w_1$ in this equation, one can assume that $|v_1 \cdots v_n| \geq |w_1 \cdots w_m|$, that is, $|v_1 \cdots v_n| \geq \frac{1}{2}|u_1 \cdots u_\ell|$.

An application of exFW theorem II

Proof.

Since $\ell \geq 6$, $|v_1 \cdots v_n| \geq \frac{1}{2}|u_1 \cdots u_\ell| \geq |u_1 u_2 u_3|$. Also note that $v_1 \cdots v_n$ is a prefix of $u_1 \cdots u_\ell$. These mean that $u_1 \cdots u_\ell$ and $v_1 \cdots v_n$ share a prefix of length

$$\max(3|u|, 3|v|) \geq 2\max(|u|, |v|) + \min(|u|, |v|).$$

Thus the exFW theorem implies that $u, v \in \{t, \theta(t)\}^+$ for some θ-primitive word $t \in \Sigma^+$. With this, the exLS equation gives $w_1 \cdots w_m \in \{t, \theta(t)\}^+$. Since t is θ-primitive, we can conclude that $w \in \{t, \theta(t)\}^+$. □

An application of exFW theorem III

This proof technique works for the cases $(5, \geq 5, \geq 5)$.

Proposition

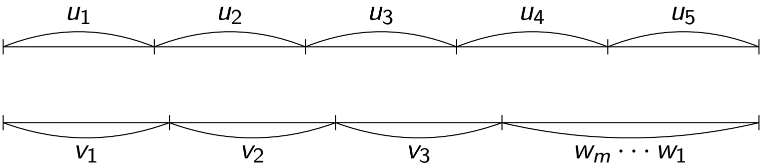
$(5, \geq 5, \geq 5)$ *imposes* θ*-periodicity.*

The remaining cases $(5, 3 \text{ or } 4, \geq 3)$ require combinatorial arguments.

Combinatorial argument on $(5, 3, \geq 3)$ I

Proposition

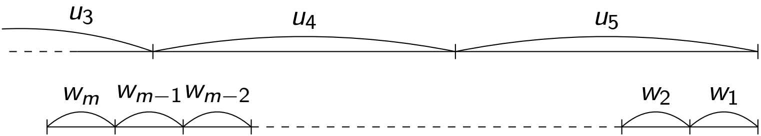
$(5, 3$ *or* $4, \geq 3)$ *impose* θ*-periodicity.*



If the border between $v_1 v_2 v_3$ and $w_m \cdots w_1$ is on anything but u_3, then the exFW theorem is still applicable.

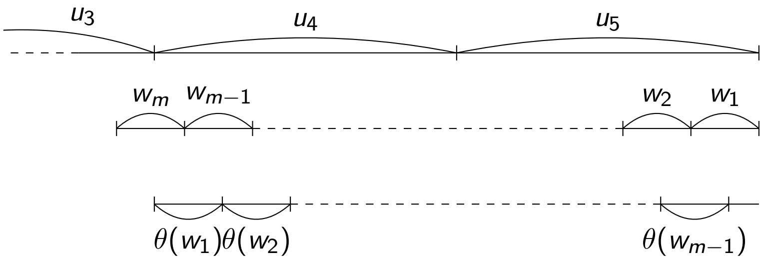
Combinatorial argument on $(5, 3, \geq 3)$ II

Even when the border is on u_3, when $|u_4 u_5| \leq |w_{m-1} \cdots w_1|$, then the exFW theorem is applicable.

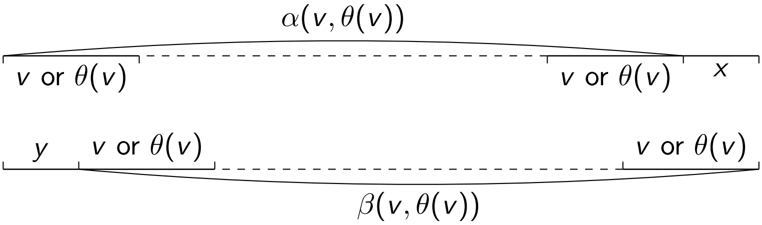


Combinatorial argument on $(5, 3, \geq 3)$ III

The case when the border is on u_3 and $u_5 = \theta(u_4)$ is illustrated.



Combinatorial argument on $(5, 3, \geq 3)$ IV



For a θ-primitive word v, we consider the overlap of the form

$$\alpha(v, \theta(v)) \cdot x = y \cdot \beta(v, \theta(v)),$$

where $\alpha(v, \theta(v)), \beta(v, \theta(v)) \in \{v, \theta(v)\}^+$, $x, y \in \Sigma^+$, and $|x|, |y| < |v|$.

Combinatorial argument on $(5, 3, \geq 3)$ V

Theorem

All possible overlaps of the above-mentioned form are given in the following table (modulo a substitution of v by $\theta(v)$) together with the characterization of their sets of solutions.

Equation	Solution
$v^i x = y\theta(v)^i, i \geq 1$	$v = yp, x = \theta(y), p = \theta(p)$, and whenever $i \geq 2$, $y = \theta(y)$
$vx = yv$	$v = (pq)^{j+1}p$, $x = qp$, $y = pq$ for some $p, q \in \Sigma^+$, $j \geq 0$
$v\theta(v)x = yv\theta(v)$,	$v = (pq)^{j+1}p$, $x = \theta(pq)$, $y = pq$, with $j \geq 0$, $qp = \theta(qp)$
$v^{i+1}x = y\theta(v)^i v, i \geq 1$	$v = r(tr)^{n+m}r(tr)^n$, $x = (tr)^m r(tr)^n$, $y = r(tr)^{n+m}$
$v\theta(v)^i x = yv^{i+1}$, $i \geq 1$	$v = (rt)^n r(rt)^{m+n}r$, $y = (rt)^n r(rt)^m$, $x = (rt)^{m+n}r$
$v\theta(v)^i x = yv^i\theta(v)$, $i \geq 2$	$v = (rt)^n r(rt)^{m+n}r$, $y = (rt)^n r(rt)^m$, $x = (tr)^m r(tr)^n$

Combinatorial argument on $(5, 3, \geq 3)$ VI

Theorem

$(5, \geq 3, \geq 3)$ *imposes* θ*-periodicity.*

Summary on the exLS equation

ℓ	n	m	θ-periodicity	How to prove
≥ 6	≥ 3	≥ 3	YES	exFW theorem
5	≥ 5	≥ 5	YES	exFW theorem
5	4	≥ 4	YES	combinatorial arguments
5	3	≥ 3	YES	
4	≥ 3	≥ 3	OPEN	?
3	≥ 3	≥ 3	OPEN	
2			NO	examples
	2		NO	
		2	NO	

Future directions

1. Solving the open cases of exLS equation;
2. Concise proof technique;
3. Further extensions of exLS equation; e.g., to n words of

$$u_1 \cdots u_\ell \in \{v_1, \theta(v_1)\}^{k_1}\{v_2, \theta(v_2)\}^{k_2} \cdots \{v_n, \theta(v_n)\}^{k_n}$$

References I

[ChKa97] Christian Choffrut and Juhani Karhumäki.
Combinatorics of words.
In Crzegorz Rozenberg and Arto Salomaa, editors, *Handbook of Formal Languages*, vol.1, pp.320-438. Springer-Verlag, Berlin-Heidelberg-NewYork, 1997.

[CKS08] E. Czeizler, L. Kari, and S. Seki.
On a special class of primitive words.
In *Proc. MFCS 2008*, LNCS 5162, pp. 265-277, Berlin-Heidelberg, 2008, Springer.

[FiWi65] N. J. Fine and H. S. Wilf.
Uniqueness theorem for periodic functions.
Proc. American Mathematical Society, 16:109-114, 1965.

References II

[HaNo04] T. Harju and D. Nowotka.
The equation $x^i = y^j z^k$ in a free semigroup.
Semigroup Forum, 68:488-490, 2004.

[Lot83] M. Lothaire.
Combinatorics on Words.
Encyclopedia of Mathematics and its Applications, 17, Addison-Wesley Publishing Co., 1983.

[LySc62] R. C. Lyndon and M. P. Schützenberger.
The equation $a^M = b^N c^P$ in a free group.
Michigan Mathematical Journal, 9:289-298, 1962.